# Signaling in social networks

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**Abstract**

I present a strategic underpinning for a process of social network formation proposed by Jackson and Rogers (2007). This paper considers cooperation between agents in a social network where agents play a high or low stake prisoner's dilemma game with each of their friends during each period. The key insight of the model is that the types of friendships an agent forms is a way to signal an agent's willingness to engage in cooperative play in the prisoner's dilemma games to follow. Signaling is valuable because it allows a cooperative type to avoid initially playing low stakes games which screen out uncooperative agents. The model provides a number of testable hypotheses about the correlation between characteristics of the social networks and the strategic environment and provides a framework for considering how different types of policies may change the social network itself.

**Keywords**: Social Networks, Signaling, Network Formation, Repeated Games, Trust.

**JEL Classification Codes:** D85, D82, A14, C72, C73.

## 1 Introduction

Social networks are important in a variety of environments. The importance of these networks has been highlighted theoretically and empirically in many areas including risk sharing (Fafchamps and Lund (2003); Bloch, Genicot and Ray (2005)), diffusion of innovation (Young 2000), adoption/diffusion of behaviour (Glaeser, Sacerdote and Scheinkman (1996); Jackson and Yuriev (2006)), trust and social capital (Mobius and Szeidl (2006)) and search in labour markets (Calvo-Armengol and Jackson (2004)). These models, for the most part,

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have treated the network structure as exogeneous and studied how in these various environments the characteristics of any given network affect individual and social welfare. Typically these papers have implications for policy for improving aggregate welfare in the form of changes to the social network. Models of network formation are needed for designing policies and assessing their likely effects.

In this paper I present a model of cooperation in a social network which is changing over time through the addition of new players. In the model players influence who they become friends with by how they search for friends, either through people they have previously met or at random from the population. In equilibrium the way in which players search for and then form friendships reveals how willing they are to engage in cooperation with a potential friend. The existence of this separating equilibrium relies on two attributes of the social network. First individuals typically have only local knowledge of the network structure, in the model an individual will know who his or her friends are and whether any of their friends know one another. Second the social network can facilitate communication between players about the historic.behavior of others. Importantly this communication need only be local in that players may only pass a message about a third player whom they both know.

In the model, a friendship allows individuals to interact in two ways. Firstly the two individuals may derive utility from the friendship by engaging in a low or high stakes prisoner's dilemma game during each period. Secondly individuals may also communicate with one another, in particular, it allows either individual to pass a warning to the other if a friend they share in common has not cooperated during an earlier prisoner's dilemma game. There are two types of people in the population one type which is prepared to cooperate and another who is not, these are modelled through the discount factor whereby cooperating individuals have a high discount factor and individuals who do not cooperate have a low discount factor. The equilibrium is semi-separating whereby cooperative (patient) agents are able to signal their type through choosing to become friends with people who know one another. The threat of communication between people who know one another means that uncooperative (impatient) agents do not choose to form friendships in the same way. This signaling results in people trusting each other more when they share a friend in common.

The understanding of how and why social networks form has come from two different strands of literature. The random graph literature and the economic literature. The former employs statistical tools to describe the construction of a network according to some mechanical algorithm and the subsequent properties of the resulting network. This strand of literature has been successful in showing how characteristics of observed networks can result from some elementary stochastic or mechanical process. What it often does not provide is a strategic underpinning for why the network forms through a particular process and not another. This is especially important in the case of social networks when individuals are making choices about how to find and become friends with other individuals. In a recent paper Jackson and Rogers (JR) (2007) show how a process of network growth, that incorporates random and network (meeting friends of friends) based meetings, produces networks which exhibit many of the stylized characteristics associated with social networks as the parameters describing the formation process are changed. On the other hand the economic literature explains network structure from a game theoretic perspective. This literature allows links to form in a network at the discretion of economic agents who are or control the nodes of the network. It explicitly incorporates costs and benefits of network formation to agents and is able to characterize a network as an equilibrium in agents' actions. This allows one to analyse the networks which form in terms of efficiency and welfare properties. By explicitly incorporating agents' decisions this literature gives us a good deal of information about what networks are stable but has thus far generally stopped short of giving predictions about degree distributions, clustering coefficients etc that one can match up to observed data.

This paper is related to both literatures in that it proposes a strategic foundation for the process of network formation proposed by JR. JR's propose an algorithm of network formation where the network of individuals grows over time through the addition of agents who connect to the existing network in a particular fashion. In particular new agents connect to existing agents in two steps. In the first step agents randomly connect to individuals in the network and in the second step agents meet friends of those individuals meet during the first step. The model presented in the current paper proposes a strategic rationale for why it is valuable to an incoming agent to form friendships with friends of friends in this second step as opposed to just anyone. The motivation for the model comes from the observation that people generally "trust" people with whom they share friends in common. In the model "trust" arises endogenously in that when two people share a friend in common each believes with probability 1 that the other will cooperate in a repeated prisoners dilemma and hence will be prepared to play a game for higher stakes. The equilibrium of the model is one in which players signal that they are willing to cooperate in a repeated prisoner's dilemma through their choice of friends.

The model illustrates how patient players can avoid an initial period of screening (playing a low stakes game), when establishing cooperation with another patient player, by sharing a friend in common with that player. The key to achieving this is that the social network facilitates communication among connected agents as well as the prospect to cooperate. In this environment impatient types (who prefer to always defect) would prefer to defect against multiple players playing in a low stakes game than successfully defect once in a high stakes game and have all their other friends find out. It is this threat of communication which allows patient players to credibly signal their type. Patient players prefer to cooperate, and are thus unconcerned of the threat posed by communication, so can credibly signal their type by choosing friends who know one another.

The purpose of this paper is to propose a model which firstly provides a motivation for why individuals may actively search for and choose to become friends with certain people (friends of friends) within the social network and secondly to use this model to explain how observed characteristics of social networks may change in different environments or change as a result of some policy. Thus far there are few papers which have been able to do both. One exception is Currarini, Jackson and Pin (2008) which derives a model of network formation to describe segregation patterns in a population of heterogenous groups of agents. In this paper the focus is primarily on the distribution of friendships and clustering aspects of a homogenous population whereas Currarini, Jackson and Pin (2008) focus on the differences between the various heterogenous groups in the population in terms of the number of friendships formed, the relative number of same-type versus other-type friends and the relative same type bias.

# 2 Model

Let time be denoted by $t = 0, ..., \infty$.

## 2.1 Social network

In period $t = 0$ there is a social network with $N$ players. This network is represented by an $N \times N$ matrix $G^0$ where an element of $G^0$, $g_{ij}^0 = 1$ indicates that player $i$ has established a friendship with player $j$. Thus the social network is directed.

Each period a new player is added to the social network and forms links with the existing players. Each successive network is described by an $(N+t) \times (N+t)$ matrix $G^t$.

Players have some knowledge of their local area of the network. Define $Q_i^t$ the neighbourhood of player $i$ at time $t$ where $Q_i^t = \left\{j : \max\left\{g_{ij}^t, g_{ji}^t\right\} = 1\right\}$. Let $K_i^t, K_{ij}^t, K_{ijk}^t$ be $K_i^t = \left\{(k,k') : \left(\max\left\{g_{kk'}^t, g_{k'k}^t\right\} = 1\right), \left((k,k') \in Q_i^t \times Q_i^t\right)\right\}$. A player $i$ at time $t$ knows $\cup_{N-i \leq t' \leq t} Q_i^t$ and $\cup_{N-i \leq t' \leq t} K_i^t$.

These assumptions are simply that a player knows the identity of his/her friends and furthermore if any of his/her friends know one another and the history of the existence of these relationships since she was born.

### 2.1.1 Prisoner's Dilemma

If a friendship exists between two players ($\max\left\{g_{ij}^t, g_{ji}^t\right\} = 1$) these players participate in a high or low stakes $s_{ij}^t = H, L$ prisoners dilemma (PD) game during each period $t =$

$0, 1...\infty$ of play. The stakes of the games a player $i$ participates in during period $t$ is $S_i^t = \left\{ s_{ij}^t | j \in Q_i^t \right\}$. The stake of the game is chosen by one of the players which I assume to be $\min \{i, j\}$ which is the older of the two players. A player $i$ then chooses an action $a_{ij}^t = \{C, D\}$ when facing player $j$ during period $t$, the set of actions a player uses during a period $t$ is given by $A_i^t = \left\{ a_{ij}^t | j \in Q_i^t \right\}$. The row player payoffs for this game are:

| | C | D |
|---|---|---|
| C | $z_s$ | $-y_s$ |
| D | $x_s$ | 0 |

The payoffs of the game are all increasing in the stake and the dominant strategy of the one shot game is to play $D$ ($x_s > z_s$ and $y_s > x_s > 0$). Also define $\Delta z = z_H - z_L$ $\Delta x = x_H - x_L$ and $\Delta y = y_H - y_L$.

### 2.1.2 Communication

A friendship also allows two players to communicate with one another about the behavior of players whom they both know for $i$ and $j$ this set of players is $Q_i^t \cap Q_j^t$. In the context of this model I limit communication to player $j$ sending player $i$ a verifiable message about any player $k \in Q_i^t \cap Q_j^t$ reporting $w_k^{ijt} = 1$ if player $k$ played $D$ when $j$ played $C$ during period $t$. I assume that upon receiving this information a player can then pass it on to any other friend $i' \in Q_i^t$ that also knows $k$, $k \in Q_i^t \cap Q_{i'}^t$. The set of communications a player $i$ receives in period $t$ is denoted by $W_i^t$.

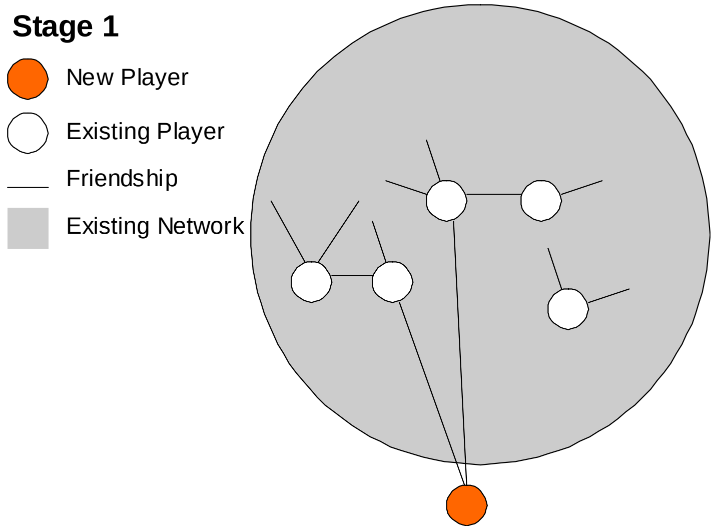
### 2.1.3 Types of players

There are two types of players patient and impatient. The discount factor $\delta$ of patient players is $\delta_P \approx 1$ and for impatient players $\delta_I \approx 0$.

### 2.1.4 Choice of friends

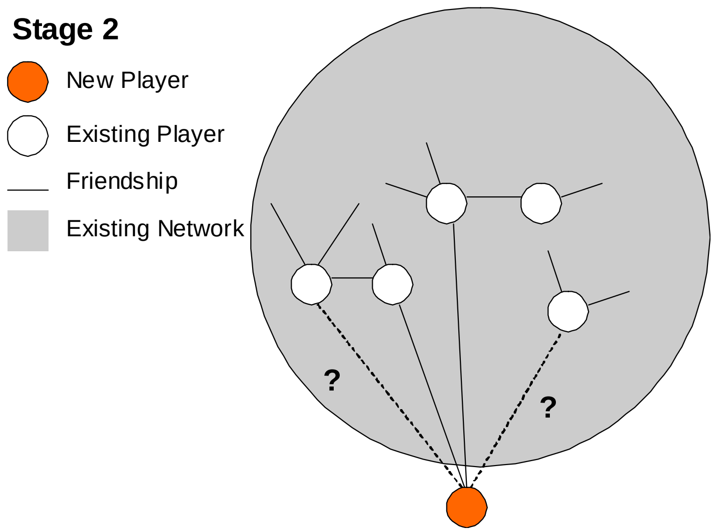
Each period a new player is born and enters the social network. When a player enters the social network she forms $M$ links with the existing network. A proportion $\gamma \in \left[0, \frac{1}{2}\right]$ of these connections are chosen by the player and the remaining $(1 - \gamma) M$ connections are established at random. For simplicity imagine the following two stage procedure:

**Stage 1** A player joining the network, randomly meets $(1 - \gamma) M$ individuals. The picture below shows stage one for the parameter values $M = 3$ and $\gamma = \frac{1}{3}$ :




**Stage 2** In stage 2 the player has a choice. A player can either connect to $\gamma M$ additional individuals, where each one of these additional individuals is a friend of one of the $(1-\gamma)M$ individuals met in stage 1 (network-based meetings); or, alternatively, a player can connect to $\gamma M$ randomly chosen individuals from the population (who may or may not be friends with the original $(1-\gamma)M$ individuals, again I assume that if they are met randomly the probability that any two are themselves friends is approximately 0) (random meetings). I assume that if player $j$ was met during stage 1 amongst his/her friends only those for whom $g_{jk}=1$ (not those for whom $g_{kj}=1$[1]) can be met in stage 2 each with equal probability $\left(\frac{1}{M}\right)$ if the player chooses to connect through network-based meetings. Below the diagram shows a new player with the choice of connecting to a friend of a friend or another randomly chosen individual.

[1] A similar assumption is needed in JR so that the alogirthm of network formation is sufficiently tractible to derive the characteristics of the social network and is maintained here for the same reason.




The existing players in the network do not observe whether or not it was possible for the new player to choose to connect to one of their friends, they only know that the new player will have the opportunity to do this with $\gamma M$ individuals out of the $(1-\gamma)M$ original individuals that the new player initially met randomly. In particular they don't know if they were part of the $\gamma M$ players whose friends the new player could have chosen to connect to. Therefore, when an existing player meets a new player and that new player does not also connect to one of her friends, she does not know whether the new player chose not to do so, or whether she never had the chance to. Looking ahead in the model patient types can signal their type by choosing to connect to $\gamma M$ friends of some of the people she met in the network, while short lived types will choose to connect to random individuals.

Finally I assume that a player with no connections is removed from the social network. This allows one to focus on the signaling incentives of new players when choosing whether or not to search for friends or friends. If impatient players remain in the social network then by searching for friends of friends this has an additional benefit of potentially identifying individuals who do not have any friends and are therefore an impatient type. This may be a realistic reason for doing so however here I would like to focus on the signaling achieved through the choice of friends.[2] By removing these impatient types, a new player believes that when it is born and forms friendships any player it finds is patient. If the new player is patient she is therefore only concerned about convincing their friends of this through their choice of random versus network based meetings.

[2] Other than re-inforcing search effort the equilibrium described in the following sections is unaffected in a setting where individuals are not removed. The social network would of course be different in the sense that impatient individuals would all be disconnected.

#### 2.1.5 Sequence of play

There is an infinite number of periods of play. During each period, play proceeds as follows:

1. A new player $i = N + t$ is born and connects to the network as described above, and all players $j < i$ update their knowledge of the network $Q_j^t, K_j^t$
2. Links are ordered randomly.
3. On the first link the oldest player $\min\{j, k\}$ (of the two, one at either end) chooses whether to play a game for high or low stakes these players update $S_i^t$.
4. The prisoner's dilemma stage game is then played, players update $A_i^t$.
5. Communication takes place.
6. Repeat steps 3-5 for the second, third ... links until every game has taken place.

Players update their personel history $h_i^t = \cup_{N-i \leq t' \leq t} \left( A_i^{t'}, S_i^{t'}, W_i^{t'}, Q_i^{t'}, K_i^{t'} \right)$ as events occur.

## 3 A sequential equilibrium of the model

In this section I present a sequential equilibrium of the model which is semi-separating of the players. In the equilibrium patient players signal their type through choosing network-based meetings during stage 2 of the joining process. In the equilibrium a patient player is able to gain the trust of a proportion $\gamma$ of friends by choosing to share friends in common. This proportion $\gamma$ of the friends then believe the player is patient with probability $\approx 1$ and is willing to cooperate and therefore agree to play a game for high stakes in the first period of interaction.

**Theorem 1** *Suppose $x_H + z_H < 2x_L$, then $\exists\ \underline{\delta} < 1$ and $\overline{\delta} > 0$ such that if $\delta_P \in (\underline{\delta}, 1)$ and $\delta_I \in (0, \overline{\delta})$ there exists a sequential equilibrium in which patient players choose network based meetings, impatient players choose random meetings and an existing player $i$ has beliefs, upon meeting, a new player $j$ $\lim_{t\to\infty} \Pr(\delta_j = \delta_P | Q_i^t \cap Q_j^t \neq \varnothing) = 1$.*

I will relegate the proof and the description of equilibrium beliefs and strategies to an appendix. Instead I will give a sketch of the equilibrium and intuition for the result.

When two players meet and share a friend in common they play a high stakes game from the first period of interaction. If the two players do not share a friend in common then a low stakes game is played in the first period and if cooperation occurs a high stakes game is played thereafter. In the equilibrium patient players choose to make network based connections when joining the network to avoid playing a low stakes game in the first period of interaction. On the other hand impatient players choose to make random connections when joining the network and defect in the low stakes game offered during the first period of interaction in all the friendships which they form. The threat of communication between two players who know one another prevents impatient types making network connections.

The choice of how to form friendships by incoming players results in existing players in the network holding beliefs $\Pr(\text{Patient}) \approx 1$ when they meet a player with whom they share a friend in common and beliefs $\Pr(\text{Impatient}) = \frac{\lambda}{\lambda+(1-\lambda)(1-2\gamma)}$ when they do not. Players update their beliefs to $\Pr(\text{Patient}) = 1$ if a player cooperates in the first period of interaction. If a player receives a warning about another player then they will update their beliefs to $\Pr(\text{Impatient}|\text{warning}) = 1$.

A patient player chooses to search for network based connections because it signals to the other player that they are patient. This allows the patient player to avoid playing a low stakes game with the players with whom they share a friend in common. An existing patient player will choose to cooperate with a player when they do not share a friend in common provided:

$$z_L + \frac{\delta_P z_H}{1-\delta_P} - \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)}\left(y_L - z_L - \frac{\delta_P z_H}{1-\delta_P}\right) > 0$$

which will be the case for $\delta_P$ sufficiently close to 1.

Impatient players choose random based connections because this type of connection prevents communication between two players who will play the impatient player. In the event that the impatient player did make a network based meeting the best she could do during the subsequent period of game play would be to cooperate with the first of the two players and defect on the second. Mimicking the patient players' friendship strategy is not better than the strategy of impatient types when defecting on two opponents in a low stakes game is better than cooperating and defecting in a high stakes game. This is the condition in the theorem, $2x_L > x_H + z_H$. Impatient types will always defect on equilibrium when they are sufficiently impatient. Off equilibrium for a sufficiently large number of people $\zeta^* > M$ who all know the impatient player and each other, the impatient player will in fact choose to cooperate because the threat of punishment facilitated by communication amongst this entire group is very large. Of course the impatient player would have to cooperate for a long period of time to acquire this number of friends so it does not occur on equilibrium.

The sequential equilibrium is one in which patient players can signal their type through sharing friends in common with their friends. The intuition for the result is that the threat to impatient players of communication and group sanctioning provided by the social network allows patient players to credibly signal their own type. This signaling motive influences the patient players pattern of friendships which they choose and is revealed in the characteristics of the network itself such as degree distributions and clustering coefficient.

The incentives which influence a player's choice of friends have implications for the dynamic process of network growth. The incentives which govern the choices of the patient players (who are the only types which remain in the network beyond the age of 1 period) determine the evolution of the social network. I consider the implications of the model as the network becomes very large. The growth process that results as an equilibrium is such that all players have an equal probability of getting a random link and a probability proportional to their in-degree of receiving network based links.

**Corollary 1** *In each period the probability an existing node has of obtaining a new link to a patient player is approximately* $(1-\lambda)\left(\frac{(1-\gamma)M}{t+N}+\frac{d_i(t)\gamma}{t+N}\right)$[3]

Proof:

Note

$$\begin{aligned} & \Pr(\text{New link from a patient player}) \\ = \; & \Pr(\text{Patient player}) \times \Pr(\text{New link}|\text{Patient player}). \end{aligned}$$

Now

$$\Pr(\text{Patient player}) = (1-\lambda);$$

and

$$\Pr(\text{New link}|\text{Patient player}) \approx \frac{(1-\gamma)M}{t+N} + \frac{(1-\gamma)Md_i(t)}{t+N}\frac{\frac{\gamma M}{(1-\gamma)M}}{M}$$

where the first term is the probability of being selected at random and the second term is the probability that a new player chooses to link to the node. The second term consists of two parts, the first is $\frac{(1-\gamma)Md_i(t)}{t+N}$ which is the probability that one of the node's neighbours is found at random and the second part is $\frac{\frac{\gamma M}{(1-\gamma)M}}{M}$ which is the probability that the node is then chosen to be linked to. This then simplifies to $\frac{(1-\gamma)M}{t+N}+\frac{d_i(t)\gamma}{t+N}$.

*QED*

The growth process specified here has the same characteristics of network formation as JR. That is all individuals can meet new friends through two channels: randomly meeting people which occurs with equal probability (independent of the number of friends) over

[3] The probablity in the theorem is approximate because it ignores the possibility some of the randomly met players are in each others' neighbourhood, or that a player could be met more than once. It is very accurate when we assume that the network is large compared to the out-degree of players $N >> M$ because the adjustments for these eventualities go to 0.

all individuals; and meeting people through other friends which occurs with a probability proportional to the number of friends the player has. This combination of randomness and network based meetings which enables JR to explain many of the stylised facts about social networks and arises, in this model, as a consequence of people trusting others with whom they share friends in common.

The process described in JR incorporates several parameters to give it the flexibility to be fitted to data on a wide variety of existing social and physical networks. JR allow four parameters $m_R, p_R, m_N, p_N$ where $m_i$ is the number of friends identified in each stage of the joining process ($R$ corresponds to a random meeting in stage 1 and $N$ corresponds to a network meeting made during stage 2) and $p_i$ is the probability each of the people met during stages 1 and 2 then becomes a friend. The equilibrium presented here encompasses the case where $p_R = p_N = 1 - \lambda$ and $(1-\gamma) M \geq \gamma M$.

# 4 Characteristics of the social network

In this section I present how the distribution of friendships and amount of clustering, in the social network which forms in the equilibrium of the previous section, relative to the primitives of the micro-foundation in this paper. This is done to illustrate how the primitives of the model can affect these characteristics.

## 4.1 Distribution of friendships

To derive characteristics of the underlying social structure I now ignore the entry of low types because these do not survive in the social network longer than 1 period. Denote time by $t$ and a player which enters at time $t$ by $i = t$. I rescale the time intervals from the previous model so a high type enters in each time period $t$ and ignore the entry of low types since they do not survive in the social network. The probability that an existing node $i$ with in-degree $d_i(t)$ gets a new link (in the next period when a High type enters) is approximately:

$$\frac{(1-\gamma) M}{t+N} + \frac{d_i(t)\gamma}{t+N}$$

**Theorem 2** *The degree distribution from a mean field approximation of the network formation process is* $\lim_{t\to\infty} F_t(d) = 1 - \left(\frac{M}{\frac{\gamma}{1-\gamma}d + M}\right)^{\frac{1}{\gamma}}$

Proof:

JR provide a proof that a process where the degree of a node born at time $i$ has initial degree $d_0$ and evolves according to

$$\frac{dd_i(t)}{dt} = \frac{ad_i(t)}{t} + \frac{b}{t} + c$$

when $a > 0$ and $c = 0$ or $a \neq 1$, then the complementary cdf is

$$1 - F_t(d) = \left( \frac{d_0 + \frac{d}{a} - \frac{ct}{1-a}}{d + \frac{b}{a} - \frac{ct}{1-a}} \right)^{1/a}$$

In the setting of this paper $d_0 = 0$, $a = \gamma$, $b = (1-\gamma) M$ and $c = 0$. So treating this as a continuous process then we have the differential equation:

$$\frac{dd_i(t)}{dt} = \frac{d_i(t)\gamma}{t+N} + \frac{(1-\gamma) M}{t+N}$$

we can solve this equation to get:

$$d_i(t) = \frac{1-\gamma}{\gamma} M \left( \frac{t+N}{i} \right)^{\gamma} - \frac{1-\gamma}{\gamma} M$$

At time $t$ $1 - F_t(d)$ is the fraction of individuals with in-degree greater than $d$. If we solve the above expression for $i$ such that $d_i(t) = d$ this then corresponds to the number of individuals who have a greater in-degree than $d$. If $i^*(d)$ is such that $d_{i^*(d)}(d) = d$ then

$$1 - F_t(d) = \frac{i^*(d)}{t+N} \text{ for all } d \text{ such that } i^*(d) > N$$

we can then derive the in-degree distribution as:

$$F_t(d) = 1 - \left( \frac{M}{\frac{\gamma}{1-\gamma} d + M} \right)^{\frac{1}{\gamma}} \text{ for all } d \text{ such that } i^*(d) > N$$

The fraction of individuals not described by this distribution $\frac{N}{t+N} \to 0$ as $t \to \infty$.

*QED*

To see how this relates to a scale free distribution we write this as the complimentary cdf:

$$1 - F_t(d) = \left( \frac{M}{\frac{\gamma}{1-\gamma} d + M} \right)^{\frac{1}{\gamma}}$$

If I now take logs of both sides we can see that this exhibits scale free properties for $d$'s which are large relative to $\frac{1-\gamma}{\gamma}M$

$$\log\left(1-F_t\left(d\right)\right)=\frac{1}{\gamma}\left[\log\left(M\right)-\log\left(\frac{\gamma}{1-\gamma}d+M\right)\right].$$

The most important property to note is that decreasing $\gamma$ results in a second order stochastic dominant shift in the degree distribution. Intuitively this is because it puts additional weight on the network meetings process which in turn biases the probability of gaining an additional connection towards those with more existing connections. This has the effect of spreading out the distribution, giving it fatter tails, relative to a distribution derived from purely random meetings.

## 4.2 Clustering

I present results for three common measures of clustering. The first is the fraction of "transitive triples." This represents the fractions of times in a network where given that $i$ knows $j$ and $j$ knows $k$ that then $i$ also knows $k$. The fraction is given by

$$C^{TT}\left(g\right)=\frac{\sum_{i;j\neq i;k\neq j,i}g_{ij}g_{jk}g_{ik}}{\sum_{i;j\neq i;k\neq j,i}g_{ij}g_{jk}}.$$

A second standard measure ignores the directed nature of the above relationships between individuals. This is a setting in which $\widehat{g}_{ij}=\max\left\{g_{ij},g_{ji}\right\}$. A measure where only the existence of the relationship rather than the directed nature of it is important is

$$C\left(g\right)=\frac{\sum_{i;j\neq i;k\neq j,i}\widehat{g}_{ij}\widehat{g}_{jk}\widehat{g}_{ik}}{\sum_{i;j\neq i;k\neq j,i}\widehat{g}_{ij}\widehat{g}_{jk}}.$$

A further variation is one in which the above $C\left(g\right)$ is calculated on a node by node basis and the average is taken across all nodes. This measure is calculated as

$$C^{Avg}\left(g\right)=\frac{1}{n}\sum_{i}\frac{\sum_{i;j\neq i;k\neq j,i}\widehat{g}_{ij}\widehat{g}_{jk}\widehat{g}_{ik}}{\sum_{i;j\neq i;k\neq j,i}\widehat{g}_{ij}\widehat{g}_{jk}}.$$

This puts relatively less weight on nodes with high degrees and more weight on low degree nodes compared with the first two measures.

**Theorem 3** *The Fraction of Transitive Triples, $C^{TT}\left(g\right)$ tends to:*

$$\frac{\gamma}{M}.$$

*Total Clustering, $C(g)$ tends to:*

$$\frac{6\gamma^2}{5M - 2 - 2\gamma(4M-2)}$$

*and Average Clustering, $C^{Avg}(g)$ tends to:*

$$\int_0^\infty \frac{1}{1-\gamma} M^{\frac{1}{\gamma}} \left(\frac{\gamma}{1-\gamma} d + M\right)^{-\frac{1+\gamma}{\gamma}} \times \left(\frac{\left(2\gamma M + 4\left(\frac{1-2\gamma}{2\gamma}\right)\left(\frac{1-\gamma}{\gamma}\right) M \log\left(\frac{d(1-\gamma)M}{\gamma} + 1\right) + 2d\right)}{(d+M)(d+M-1)}\right) dd$$

*Proof:*

The growth process which describes this model is for a subset of possible parameter values from the process in JR.

JR prove that $C^{TT}$ tends to:

$$\frac{p_R}{m(1+r)}$$

if $\frac{p_R}{r} \leq 1$. $C(g)$ tends to:

$$\frac{6p_R}{(1+r)\left[(3m-2)(r-1)+2mr\right]}$$

if $r > 1$. $C^{\text{Avg}}(g)$ tends to:

$$\int_0^\infty \left[\frac{(rm)^{r+1}(r+1)}{(d+rm)^{r+2}}\right] \left(\frac{1}{(d+M)(d+M-1)/2}\right) \times \left(\begin{array}{c} m^2 C^{TT}\left(1 + \frac{2d(1+r)}{m}\right) - p_R d \\ + rm\left[\log\left(\frac{d}{rm}+1\right)\right]\left(\frac{p_R}{r} + p_R - 2C^{TT} m(1+r)\right) \end{array}\right) dd$$

In terms of the parameters in this model $p_R = 1$ $m = M$ and $r = \frac{1-\gamma}{\gamma} > 1$ since $\gamma \leq \frac{1}{2}$. Making these substitutions the result follows.

*QED*

The most important property to observe is that the first two measures of clustering $C^{TT}$ and $C$ are monotone increasing in the number of network based connections $\gamma$. The intuition for this is that increasing $\gamma$ increases the proportion of people met during stage one for whom an additional friend is met which increases the number of triads (three people who all know one another) in the network.

## 5 Number of non-random connections

In this section I extend the model to endogenize the new player's search process for friends further. In the earlier model both patient and impatient players had strict incentives to choose either network based meetings in the case of patient players or random meetings in the case of impatient players. In this section, I allow players to expend costly effort to affect the relative number of network to random meetings. One can think of this as effort to attend the same parties, venues, clubs, etc. of the individuals they have already met, the people they then meet at these events are more likely to share a friend in common than similar events which aren't attended by anyone they know.

I assume an individual may expend costly effort $e$ to affect $\gamma(e)$ the fraction of network based meetings. To simplify notation I will make $\gamma$ the choice variable for the individual and denote the cost of effort as $C(\gamma)$. I ignore integer constraints and assume this cost of effort is continuous in $\gamma$ moreover $C'(\gamma), C''(\gamma) > 0$.

In this section I define the value of signaling in terms of model primitives. I place some additional structure on payoff in particular I assume the ratio of high to low stakes payoffs are equal for $z, x, y$ $\frac{z_H}{z_L} = \frac{x_H}{x_L} = \beta$ and the ratio of cooperative payoff to successful defection payoff is equal in low and high stake games $\frac{z_L}{x_L} = \frac{z_H}{x_H} = \alpha$. In this section I focus on how $\delta_I, \delta_P, \alpha, \beta$ affect the incentives for individuals to search for friends of friends and the subsequent effect on the structure of the network which eventuates. This then provides a framework for studying the impact of proposed policies on network structure through a policies impact on this model primitives. It also allows one to formulate testable hypotheses of how network characteristics should vary across different environments. Throughout this analysis I am holding $M$ (the number of friendships a new player makes) constant.

In the equilibrium described in the previous section the value of signaling is the additional benefit a patient player receives by avoiding the period of screening $\Delta z$. For impatient players to prefer to defect initially, rather than wait one period to defect in a high stakes game, it was assumed that

$$x_L > z_L + \delta_I x_H$$

which is true for $\delta_I \to 0$. If however the discount factor of impatient types $(\delta_I)$ is too high this is no longer true. To maintain a signaling equilibrium where there is screening of impatient types the number of periods for which players play the low stakes game is increased. I assume that the existing players in the network can commit to the stake of games in future periods conditional on cooperation in prior periods. If this is the case then for an impatient type to defect in the first period the value of defecting immediately exceeds the value of waiting until a high stakes game is offered and defecting then. So the number of periods in which a low stakes game is offered must be at least large enough such that

$$x_L \geq \frac{1-\delta_I^n}{1-\delta_I} z_L + \delta_I^n x_H$$

where the value on the left is the utility from defecting immediately and the value of the right is the value from cooperating for $n$ periods before defecting in the first high stakes game. Now defining $\underline{n}$ implicitly by[4]:

$$\begin{aligned} x_L &= \frac{1-\delta_I^{\underline{n}}}{1-\delta_I} z_L + \delta_I^{\underline{n}} x_H \\ 1 &= \frac{1-\delta_I^{\underline{n}}}{1-\delta_I} \alpha + \delta_I^{\underline{n}} \beta \end{aligned}$$

then $\underline{n}'(\delta_I) > 0$ $\underline{n}'(\alpha) > 0$ $\underline{n}'(\beta) > 0$. Therefore the value of signaling $V_{\text{Signal}}$ is:

$$V_{\text{Signal}} = 2\frac{1-\delta_P^{\underline{n}}}{1-\delta_P} (\beta - 1) z_L$$

where the 2 comes from having an additional friend of a friend signals to both friends one's own type. If the marginal cost of obtaining friends of friends is convex then patient players will include as many friends of friends amongst the $M$ friendships they establish, when they are born, such that the marginal cost of doing so is less than $2\frac{1-\delta_H^{\underline{n}}}{1-\delta_H}(\beta-1) z_L$. Assuming an interior solution (this number is greater than 0 and less than $\frac{M}{2}$) then $\gamma(\delta_I, \delta_P, \alpha, \beta)$ is given by:

$$2\frac{1-\delta^{\underline{n}}}{1-\delta_P}(\beta-1) z_L = C'(\gamma M)$$

The main theorem in this section describes the comparative statics of this relationship.

**Theorem 4** *The number of network meetings a patient type chooses, $\gamma M$, is increasing in $\delta_P, \delta_I, \alpha$ and $\beta$.*

Proof:

It suffices to show $\gamma$ is increasing in $\Delta z$, $\delta_I$ and $\delta_P$. First note that $V_{\text{Signal}}$ is increasing in $\delta_I$, and $\alpha$ since:

[4] In principle the right hand side of this expression should include a term incorporating the probability of obtaining an additional friendship, which becomes increasingly likely as $\underline{n}$ becomes large. However I will assume that the frequency of the arrival of new players decreases as $\delta_L$ increases. That is in effect the rate of time preference is not changing but rather the frequency of interaction increases as $\delta$ increases.

$$\frac{d\underline{n}}{d\delta_I} > 0; \frac{d\underline{n}}{d\alpha} > 0; \frac{\partial V_{\text{Signal}}}{\partial \underline{n}} > 0$$
$$\implies \frac{\partial V_{\text{Signal}}}{\partial \underline{n}} \frac{d\underline{n}}{d\delta_I} = \frac{dV_{\text{Signal}}}{d\delta_I} > 0;$$

Also $\frac{dV_{\text{Signal}}}{d\beta} = \frac{\partial V_{\text{Signal}}}{\partial \beta} + \frac{\partial V_{\text{Signal}}}{\partial \underline{n}} \frac{d\underline{n}}{d\beta} > 0$ since

$$\frac{d\underline{n}}{d\beta} > 0; \frac{\partial V_{\text{Signal}}}{\partial \underline{n}} > 0; \frac{\partial V_{\text{Signal}}}{\partial \beta} > 0$$

Furthermore $V_{\text{Signal}}$ is increasing in $\delta_P$; Now defining:

$$F(\gamma, \Delta z, \delta_I, \delta_P) = 2\frac{1-\delta_P^{\underline{n}(\delta_I)}}{1-\delta_P}\Delta z - C'(\gamma M)$$

and noting

$$\frac{dV_{\text{Signal}}}{d\delta_I} > 0; \frac{dV_{\text{Signal}}}{d\delta_P} > 0; \frac{dV_{\text{Signal}}}{d\alpha} > 0; \frac{dV_{\text{Signal}}}{d\beta} > 0; C'' > 0$$

we can sign $\frac{d\gamma}{d\delta_I}$ by:

$$sign\left(\frac{d\gamma}{d\delta_I}\right) = -sign\left(\frac{\frac{\partial F}{\partial \delta_I}}{\frac{\partial F}{\partial \gamma}}\right) = \frac{sign\left(\frac{dV_{\text{Signal}}}{d\delta_I}\right)}{sign\left(C''(\gamma M)\right)} > 0$$

and similarly the results follow for $\frac{d\gamma}{d\delta_P}, \frac{d\gamma}{d\alpha}, \frac{d\gamma}{d\beta}$

*QED*

The theorem shows that search effort is increasing in the ratio of high versus low stakes $\beta$, the ratio of payoffs from cooperating to defecting, and the discount rates of both the patient cooperating player and the impatient non-cooperative player. This allows one to analyse how a broadbased policy designed to affect the social network will impact it through the changes it has on $\alpha$, $\beta$, $\delta_I$ and $\delta_P$. It also gives one a basis for predicting in which settings networks with higher $\gamma$'s will exist compared to others.

## 5.1 Policy Analysis

The purpose of this section is to illustrate how the signaling motive provided by the model can be used to analyze policies designed to alter the structure of a social network. Specifically we are interested in how a policy will affect a new player's choice of $\gamma$. Provided that the policy change is not so great that the conditions required for the signaling equilibrium to exist are not violated. The effect of a new policy can be inferred from its impact on $V_{\text{Signal}}$ through the underlying parameters $\alpha$, $\beta$, $\delta_P$, and $\delta_I$. If the impact is positive then from Theorem 4 the impact is also positive on $\gamma$ and the resulting social network will SOSD (Theorem 2) and have greater clustering (Theorem 3) than a social network absent the policy change.

For example in a given network consider a policy which enables more frequent interaction among agents. The effect of this can be interpretted as a shift in the discount rate of both patient and impatient agents. Now from Theorem 4 this will increase $\gamma$ which increases both the level of clustering and the fat tailed nature of the degree distribution.

### 5.2 Comparison of networks

The model also permits one to compare different social networks on the basis of the relative proportion of network meetings $\gamma$ and infer the relative value of signaling in each network. That is assuming the costs of searching are similar across networks, a network with a greater amount of network based meetings is one in which the value of signaling is greater as well. This may then allow one to make inferences about the relative frequency of interactions or the value derived from cooperation. Furthermore it gives a number of testable hypothesises about the correlation between social network characteristics (clustering coefficients and degree distributions) and a number of potentially observable parameters of relationships (frequency of interaction, benefits from cooperation, incentives for defection).

### 5.3 Welfare

#### 5.3.1 Inefficiently low network based meetings

The utilitarian social welfare maximizing level of signaling is less than optimal in equilibrium. In the model agents only incorporate one half of the benefits from signaling into their choice of meeting friends of friends. That is they ignore the benefit that the old player gets from meeting a new player through a network based meeting compared to a random meeting. The benefit to society of one additional network based meeting is in fact $2V_{\text{Signal}}$ since the new player and older player avoids a period(s) of screening.

#### 5.3.2 Trade-off between efficiency of network and equality

To the extent that inequality is a concern, increases in the efficiency of the social network through greater levels of signaling will also result in greater inequality across the society. Each additional friendship between two patient types benefits both agents and so the agents with the greatest number of friendships are also the agents that derive the greatest level of utility in the society. So from this point of view there is a trade off between improving equality in the population and improving the level of trust. It is important to note at this point that exogenously changing the network through targetted addition and/or deletion of friendships is not an instrument available to a benevolent social planner. Rather, when the social planner is restricted to manipulations to the network through the parameters affecting the value of signaling then there is necessarily a tension between improving efficiency and reducing inequality. Reducing inequality comes at the expense of efficiency and vice versa.

### 5.4 Renegotiation of the stake of the game

When two players do not share a friend in common we may be concerned that in the previous section the sequence of stakes offered may not be renegotiation proof. Specifically that after the first period, when patient types reveal themselves by cooperating, then the two players share a common belief that the other is patient. If this is the case then both players will prefer to switch to a high stakes game in the second period. However if it is not possible to commit to the sequence of stakes which will be offered then it is not an equilibrium for the impatient type to defect for sure in the first period. Rather they would strickly prefer to cooperate and renegotiate a high stakes game in the second period in which they would subsequently defect. In this section I demonstrate that there exists a renegotiation proof sequential equilibrium of the two player relationship which exhibits the qualitative features of the equilibrium in the commitment case. That is the expected payoff for the impatient new player is the same, $x_L$, the expected payoff for the patient new player is $\frac{z_L}{1-\delta_P} + \frac{\Delta z \delta_P \beta(\delta_I)}{(1-\delta_P)(1-\delta_P(1-\beta(\delta_I)))}$ and the comparative statics of $V_{\text{Signal}}$ in Theorem 5 continue to hold in the renegotiation setting.

I impliment a renegotiation procedure closely related to one used in Watson (1999) whereby the renegotiation is limited to a local escalation or de-escalation in the timing of the current regime. As is standard in the literature on renegotiation I assume players vote on whether to abandon one regime in favour of another. The renegotiation criterion considers *jump alterations* and *stall alterations*. In a *jump alteration* at a time $t$ players agree to continue the current regime as if they were continuing from a time $s > t$. In a *stall alteration* players agree to resume the current regime as planned after a delay until time $s > t$. These jumps and stalls are incentive compatible provided that the new regime is an equilibrium (incentive-compatible) continuation of the game.

**Definition 1** *An equilibrium regime is called alteration proof if for every $t \geq 0$ every incentive compatible alteration is defeated.*

For simplicity suppose we focus on a simple two player friendship where the players do not share any friends in common and one of the players can be either patient or impatient. The ex ante probability of this player being impatient is $\frac{\lambda}{[(1-2\lambda)(1-\gamma)+\lambda]}$. In this setting the only way to learn that a player is impatient is if the player defects and thereby ends the friendship. Consider the following characteristics of a conjectured equilibrium:

1. A low stakes game is offered in the first period

2. Impatient types mix in the first period between defecting and cooperating with probability $\mu$

3. In period 2 the older player mixes with probability $\beta$ between offering a high and low stakes game

4. From period 2 onwards if a high stakes game is offered the impatient type defects and if a low stakes game is offer the impatient type cooperates.

5. From period 3 onwards the older player either offers a high stakes game if a high stakes game was played previously or mixes with probability $\beta$ if only low stakes games have been offered.

**Theorem 5** *Suppose* $\lambda \geq \frac{(1-\gamma)\Delta z}{z_L+y_H(1-\delta_P)+2\Delta z(1-\gamma)}$, $\beta = \frac{1}{(x_H-x_L)}\left(\frac{(x_L-z_L)}{\delta_I} - x_L\right)$, *$\mu$ satisfies* $\frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}\frac{z_H}{1-\delta_P} + \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}(-y_H) = \frac{z_L}{1-\delta_P}$, $1-\max\left\{\frac{z_L}{x_L}, \frac{z_H}{x_H}\right\} > \delta_I > \frac{x_L-z_L}{x_H}$ *then* $\exists\underline{\delta}$ *such that there is an alteration proof equilibrium of the two person repeated prisoners dilemma for* $\delta_P \in [\underline{\delta}, 1]$ *which satisfies the above characteristics.*

I will leave the proof for the appendix but will give an informal description of why the equilibrium is renegotiation proof. It satisfies alteration proofness because, in the case of delays, this sequence of stakes is weakly increasing and neither agent would like to delay the revelation of information. In the case of jumps, the only jump which is weakly preferred from the older players point of view is a jump to playing high stakes game. This is not incentive compatible because it requires all impatient types to have defected by the previous period which is not incentive compatible for impatient types when $\delta_I > \frac{x_L-z_L}{x_H}$.

In this equilibrium the expected payoff of the impatient new player is $x_L$ because the equilibrium is such that the impatient player is indifferent between defecting in the first period and continuing with the relationship by cooperating. The payoff of the patient new player is $\frac{z_L}{1-\delta_P} + \frac{\Delta z\delta_P\beta(\delta_I)}{(1-\delta_P)(1-\delta_P(1-\beta(\delta_I)))}$ and the value of signaling is

$$\begin{aligned} V_{\text{Signal}} &= \frac{z_H}{1-\delta_P} - \frac{z_L}{1-\delta_P} - \frac{\Delta z\delta_P\beta(\delta_I)}{(1-\delta_P)(1-\delta_P(1-\beta(\delta_I)))} \\ &= \frac{\Delta z}{(1-\delta_P(1-\beta(\delta_I)))} \end{aligned}$$

where $\beta(\delta_I)$ is decreasing in $\delta_I$ so the payoff to the patient player is also decreasing in $\delta_I$ and we can conclude that the comparative statics in Theorem 5 are unchanged for $\delta_I, \delta_P$, and $\Delta z$.

I claim without proof that modifying the commitment equilibrium from the previous section, by having players play the renegotiation equilibrium presented here in the instances where players do not share a friend in common, will not qualitatively change any of the predictions of the model. The difficulty in doing this explicitly is that the values of $\beta$ and $\mu$ must necessarily include the probabilities of obtaining further friends through maintaining the friendship since despite these terms approaching zero as $N \to \infty$ $\beta$ and $\mu$ are set to keep the two players indifferent along the on-equilibrium path where low stakes games are being offered. For expositional simplicity the analysis here effectively ignores these small adjustments and presents the equilibrium of the repeated interaction when there is no prospective of gaining additional friendships.

# 6 Conclusion

This paper begins with the observation that sharing a friend in common can be valuable in establishing trust, especially early on in a relationship. It builds on this observation to develop a model of network formation as an equilibrium in individuals' actions, whereby players' believe players with whom they share a friend will be prepared to cooperate in a prisoner's dilemma game. Furthermore it shows that this equilibrium is equivalent to a process of network formation derived by JR which can explain a number of the characteristics of real world networks.

There are a number of advantages to nesting a process of network formation in an equilibrium. First it allows key characteristics of social networks, degree distributions and clustering coefficients, to be related to underlying properties of the environment. Second it provides a framework for understanding how potential policies can change the underlying social structure or more importantly how to design policies to achieve certain policy objectives through changing the social network. Finally on an empirical note it gives a number of testable hypothesises about the correlation between social network characteristics (clustering coefficients and degree distributions) and a number of potentially observable parameters of relationships (frequency of interaction, benefits from cooperation, incentives for defection, duration of friendship).

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# 7 Appendix 1 Sequential Equilibrium and Proof

Before giving the proof I will describe the strategies and beliefs which support the equilibrium. It will be useful to define a partition $\Pi_i^t$ over the friends of player $i$ at time $t$. This partition groups individuals together if a warning about player $i$ can spread from anyone of them to the rest. Define the group of player $j$ in the partition of player $i$'s neighbors by

$$\pi_i^t(j) = \{j\} \cup \left\{ \begin{array}{c} k_n | \exists k_1, ...k_n \in Q_i^t, n > 0 : \\ \max\left\{g_{jk_1}^t, g_{k_1 j}^t\right\} = 1, \max\left\{g_{k_w k_{w+1}}^t, g_{k_{w+1} k_w}^t\right\} = 1 \text{ for all } w \leq n \end{array} \right\}$$

The following strategies and beliefs constitute a sequential equilibrium.

**Patient player $i$ strategy**

- *Choice of friends* {Network,Random}
  - Player $i$ chooses Network based meetings.
- *Choice of stake* $s_{ij}^t \in \{\text{High}, \text{Low}\}$ *in period* $t$ *against a player* $j$.
  - Player $i$ specifies a high stakes game in period $t$ with player $j$:
  - $t > N + j$
    - Warnings - $w_j^{ikt'} = 0$ : for all $t' \leq t$ and $k \in \pi_i(j)$; and
    - Stakes - Stakes have been on-equlibrium for all $t' < t$; and
    - Actions - $\left(a_{ik}^{t'}, a_{ki}^{t'}\right) = (C, C)$ for all $t' < t$ and $k \in \pi_i(j)$; and
    - Connections - No changes in connections $K_{ij}^t$ not accompanied by a warning.
  - $t = N + j$
    - Warnings - $w_j^{ikt'} = 0$ : for all $t' \leq t$ and $k \in \pi_i(j)$; and
    - Stakes - No history of past stakes $S$; and
    - Actions - $\left(a_{ik}^{t'}, a_{ki}^{t'}\right) = (C, C)$ for all $t' < t$ and $k \in \pi_i(j)$; and
    - Connections - $\exists k \in Q_i^t : \max\left\{g_{jk}^t, g_{k,j}^t\right\} = 1$;no changes in connections $K_{ij}^t$ not accompanied by a warning.
  - Player $i$ chooses a low stakes game otherwise
- *Choice of strategy* $a_{ij}^t \in \{C, D\}$ *against a player* $j$ *in period* $t$.
  - Player $i$ plays C when:
    - Warnings - $w_j^{ikt'} = 0$ and $w_i^{jkt'} = 0$ : for all $t' \leq t$ and $k \in \pi_i(j)$; and
    - Stakes have been on equilibrium; and
    - Actions - $\left(a_{ij}^{t'}, a_{ji}^{t'}\right) = (C, C)$ for all $t' < t$; and
    - Connections - No changes in connections $K_{ij}^t$ $K_{ji}^t$ not accompanied by a warning.

- Player $i$ plays D otherwise.

- *Choice of sending warning about player $j$*
  - If player $j$ deviated player $i$ always sends a warning

**Impatient player $i$ strategy**

- *Choice of friends* {Network,Random}
  - Player $i$ chooses to connect to random players.

- *Choice of stake in each period $t$ against a player $j$ for which $g_{ij}^{t}=1$, $s_{ij}^{t}\in\{\text{High},\text{Low}\}$.*
  - Player $i$ specifies a high stakes game in period $t$ with player $j$:
  - $t>N+j$
    * Warnings - $w_{j}^{ikt'}=0$ and $w_{i}^{jkt'}=0$ : for all $t'\leq t$ and $k\in\pi_{i}(j)$; and
    * Past stakes have been on equlibrium; and
    * Actions - $\left(a_{ij}^{t'},a_{ji}^{t'}\right)=(C,C)$ for all $t'<t$ ; and
    * Connections - No observed changes in connections for $K_{ij}^{t}$ not accompanied by a warning.
  - $t=N+j$
    * Warnings - $w_{j}^{ikt'}=0$ and $w_{i}^{jkt'}=0$ : for all $t'\leq t$ and $k\in\pi_{i}(j)$; and
    * Stakes - No history of past stakes; and
    * Actions - No history of past actions; and
    * Connections - $\exists k\in Q_{i}^{t}$ : $\max\{g_{jk},g_{k,j}\}=1$;no changes in connections $K_{ij}^{t}$ not accompanied by a warning.
  - Player $i$ chooses a low stakes game otherwise

- *Choice of strategy $a_{ij}^{t}\in\{C,D\}$ against a player $j$ in period $t$.*
  - Play C:
    * Warnings - $w_{j}^{ikt'}=0$ and $w_{i}^{jkt'}=0$ : for all $t'\leq t$ and $k\in\pi_{i}(j)$
    * Past stakes have been on equlibrium;
    * No changes in connections $K_{ij}^{t}$ not accompanied by a warning.and
    * $i$ is yet to play $j'$ in period $t$; or
    * $|\pi_{i}(j)|>\zeta^{*}(t)>M$

- Play D otherwise

- *Choice of sending warning about player j*
  - If player $j$ deviates player $i$ always sends a warning

**Player's Beliefs**

Observing network structure:

- The belief of a player with degree $d$ at time $t$ when that player meets a new player $k$ who is not a friend of a friend (stranger) and I have not received a warning about is

$$\begin{aligned} & \Pr\left(\delta_j = \delta_I | w_j = 0; \nexists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right) \\ = & \frac{\lambda\left(1 - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right)}{\lambda\left(1 - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right) + (1-\lambda)(1-2\gamma)\left(\Pi_{i=0}^{(1-2\gamma)M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right)}. \end{aligned}$$

- where

$$\lim_{t\to\infty} \Pr\left(\delta_j = \delta_I | w_j = 0; \nexists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right) = \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)}$$

- The belief of a player with degree $d$ at time $t$ when that player meets a new player $k$ with whom a friend is shared and they have not received a warning about is:

$$\Pr\left(\delta_k = \delta_I | w_j = 0; \exists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right) = \frac{\lambda\Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)}{\lambda\Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right) + 2\gamma(1-\lambda)}.$$

  where

$$\lim_{t\to\infty} \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right) = 0$$

Receiving warnings:

- A player's belief about another player $j$ in any history when he/she receives warning about them $w_j^{ikt} = 1$:

$$\Pr_i\left(\delta_j = \delta_I | w_j^{ikt} = 1 \text{ for some } k \in Q_i^t\right) = 1$$

- A player's belief about another player $j$ in any history when he/she receives warning from them $w_k^{ijt} = 1$ :

$$\Pr_i\left(\delta_j = \delta_I | w_k^{ijt} = 1 \text{ for some } k \in Q_i^t\right) = 1$$

Observing actions:

- The belief about another player after she has cooperated at least once and there has been no warning reeived about that player

$$\Pr\left(\delta_j = \delta_I | a_{ji}^t = C \text{ for all } t \geq j - N; w_j^{ikt} = 0\right) = 0.$$

Observing choice of stakes by an older player:

- An off equilibrium increase in the stake

$$\Pr_i\left(\delta_j = \delta_I | s_{ji}^t = High; j < i; t = i; Q_i \cap Q_j = \varnothing\right) = 1$$

- An off equilibrium decrease in the stake does not affect beliefs.

Observing off equilibrium changes in $K_i^t$:

- Observing changes in $K_i^t$ without receiving a warning results in beliefs

$$\Pr\left(\delta_j = \delta_I | K_{ijk}^{t'} \neq K_{ijk}^t \text{ for some } t' \leq t; w_j^{ikt'} = 0 \text{ and } w_k^{ijt'} = 0 \text{ for all } t' \leq t\right) = 1.$$

**Proof:**
**Patient Player $i$**
*Choice of action in stage game (against player $j$)*

- Playing C occurs when:
  - * Warnings - $w_j^{ikt'} = 0$ and $w_i^{jkt'} = 0$ : for all $t' \leq t$ and $k \in \pi_i(j)$; and
    * Stakes - no off-equilibrium increase in the stake; and
    * Actions - $\left(a_{ij}^{t'}, a_{ji}^{t'}\right) = (C, C)$ for all $t' < t$; and
    * Connections - No changes in connections $K_{ij}^t$ not accompanied by a warning.

Case 1:
If $t > N - j$ then the patient players beliefs are

$$\Pr\left(\delta_j = \delta^I | a_{ji}^t = C \text{ for all } t \geq j - N; w_j^{ikt} = 0\right) = 0$$

and the game is a high stakes game. Thus provided

$$\frac{z_H}{1 - \delta_P} \geq x_H$$

is satisfied then this is the best action the agent can take.

Case 2:

If $t = N - j$ and the players share a friend in commonthe older players beliefs are

$$\Pr\left(\text{impatient}|w_k = 0; \exists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right) = \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)$$

and the game is a high stakes game. Thus provided

$$\left(1 - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right)\frac{z_H}{1-\delta_P} - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)y_H \geq \left(1 - \Pi_{i=0}^{M-1}\left(\frac{N-d-i}{N+t-1-i}\right)\right)x_H$$

is satisfied then this is the best action the agent can take. For the younger player beliefs are

$$\Pr\left(\text{impatient}|w_k = 0; \exists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1, j < i\right) = 0$$

Thus provided

$$\frac{z_H}{1-\delta_P} \geq x_H$$

is satisfied then this is the best action the agent can take.

Case 3:

If $t = N - j$ and $Q_i^t \cap Q_j^t = \varnothing$ then the older players beliefs are

$$\Pr\left(\text{impatient}|w_k = 0; \nexists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1\right) \approx \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)}$$

Thus provided

$$\left(1 - \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)}\right)\left(\frac{\delta_P z_H}{1-\delta_P} + z_L\right) - \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)}y_L \geq \left(1 - \frac{\lambda}{\lambda + (1-\lambda)(1-2\gamma)}\right)x_L$$

is satisfied then this is the best action the agent can take. If the player is the new player then the belief it has is

$$\Pr\left(\text{impatient}|w_k = 0; \exists k' \in Q_i^t : \max\left\{g_{k,k'}^t, g_{k',k}^t\right\} = 1, j < i\right) = 0$$

and the strategy is optimal provided:

$$(1-\lambda)\left(\frac{\delta_P z_H}{1-\delta_P} + z_L\right) - \lambda y_L \geq (1-\lambda)\,x_L$$

$\exists\underline{\delta}$ such that $\delta_P \in [\underline{\delta}, 1]$ satisfies all of the above conditions.

- Play D otherwise

– Means at least one of the following has occured

  * Warnings - $w_j^{ikt'} \neq 0$ or $w_i^{jkt'} \neq 0$ : for a $t' \leq t$ and $k \in \pi_i(j)$; or
  * Stakes - An off equilibrium increase in the stake;
  * Actions - either $i$ or $j$ has played $D$ previously; or
  * Connections - A change in connections $K_{ij}^t$ or $K_{ji}^t$not accompanied by a warning.

In all instances the updated beliefs of at least one of the individuals are

$$\Pr(\text{impatient}) = 1$$

and the other player knows this so the optimal action is $D$.

- *Choice of stake in game*

  – An off-equilibrium choice of a high stake game results in the new player believing $\Pr(\text{impatient}) = 1$ and in the subsequent game the new player will play defect. The most the patient player can get is therefore 0. This is not optimal since the patient player can receive at least this by choosing an on-equilibrium low stake game and an on-equilibrium action, an on-equilibrium defect also results in a payoff of 0 but an on-equilibrium cooperate has a positive expected payoff. An off-equilibrium choice of a low stakes game will not change the other players choice of action so is worse.

- *Choice of links*

  – For a patient player choosing to signal that they are patient by establishing links with friends of friends is beneficial because it avoids the low stakes screening game in the first period. Thus gains a benefit $\Delta z$ on each relationship.

- *Choice to send warning*

  – Sending a warning is costless and does not affect a player's payoffs so is always optimal.

**Impatient Player**

- *Choice of action in stage game*

  – Play C:

* Warnings - $w_j^{ikt'} = 0$ and $w_i^{jkt'} = 0$ : for all $t' \leq t$ and $k \in \pi_i(j)$
* Past stakes have been on equlibrium;
* No changes in connections $K_{ij}^t$ not accompanied by a warning; and
* $\exists j' \in \pi_i(j)$ who player $i$ is yet to play in period $t$; or
* $|\pi_i(j)| > \zeta^*(t) > M$

2 cases.

1) $\exists j' \in \pi_i(j)$ who player $i$ is yet to play in period $t$.

The times when the impatient player chooses to play $C$ against a player $j$ are when there is another $j' \in \pi_i(j)$ who would receive a warning, in which case the impatient player is better of choosing to cooperate with $j$ and then play $D$ later in the period.

$$\frac{z_H}{1-\delta_I}$$

2) The other time an impatient player $i$ will cooperate is if the shared neighbourhood $\pi_i(j)$ is sufficiently large. For any $\delta_I$ there will exist a shared neighborhood $\pi_i(j)$ such that the player will prefer to cooperate. The key to the proof is to show that the size of this neighborhood is $> M$ and will therefore never eventuate on the equilibrium path. The impatient player will destroy all of her initial friendships so even if by chance these individuals know one another the player will still choose to defect on one of them in the first period.

For any $\delta > 0$ there is a size of neighborhood $\zeta^* = |\pi_i^*(j)|$ such that the payoff from cooperating forever excedes the payoff from defecting. Ignoring the increased probability of meeting future players from maintaining connections today this would be:

$$\frac{\zeta^*(\delta)\,\delta z_H}{1-\delta} + z_H = x_H$$

including the payoffs from future meetings increases the left hand side. Hence there is a range of neighborhoods $|\pi_i(j)| > \zeta^*$ where the impatient player will prefer to cooperate with players $k \in \pi_i(j)$. The remainder of the proof is to show that $\exists\delta > 0$ such that a neighborhood of size $M$ is not sufficiently large to induce the impatient player to cooperate. This is the largest possible neighborhood an impatient player may have in the first period it is born. Given on-equilibrium actions the impatient player will not gain more connections than this because it will be removed from the network after 2 periods.

A playermay obtain future benefits from meeting incoming patient players in the future if it maintains a friendship today. This effect lowers the threshold size $\zeta^*$. Provided that the threshold is never $\leq M$ then this will never eventuate on-equilibrium. The expected number of future in-degree friendships a player will obtain over time as the population grows is:

$$\frac{dd_i(t)}{dt} = \frac{d_i(t)\gamma}{N+t} + \frac{(1-\gamma)M}{N+t}$$

with solution

$$d_i(t) = \frac{1-\gamma}{\gamma} M \left(\frac{N+t}{i}\right)^{\gamma} - \frac{1-\gamma}{\gamma} M$$

The largest number of in-links at time $t$ is therefore $\frac{1-\gamma}{\gamma} M \left(\left(1+\frac{t}{N}\right)^{\gamma} - 1\right)$

The expected profits from future connections if the player cooperates is then

$$M\frac{1-\gamma}{\gamma} z_H \sum_{t=1}^{\infty} \delta^t \left(\left(1+\frac{t}{N}\right)^{\gamma} - 1\right)$$

for $\delta, \gamma < 1$ this is finite and $\lim_{\delta \to 0} M\frac{1-\gamma}{\gamma} z_H \sum_{t=1}^{\infty} \delta^t \left(\left(1+\frac{t}{N}\right)^{\gamma} - 1\right) = 0$. Hence if an impatient player $i$ is facing a player $j$ and has $|\pi_i(j)| = M$ then payoff from cooperating is at most

$$\frac{\delta M z_H}{1-\delta} + z_H + M\frac{1-\gamma}{\gamma} z_H \sum_{t=1}^{\infty} \delta^t \left(\left(1+\frac{t}{N}\right)^{\gamma} - 1\right)$$

and in the limit as $\delta \to 0$ is $z_H$. Hence for $\exists \delta : \delta_I \in [0, \delta]$ the impatient player will prefer to play $D$.

- *Choice of stake in game*
  - When facing a new player $j$ where $Q_i \cap Q_j = \varnothing$ choosing a high stakes game instead of a low stakes game will result in

    $$\Pr_j(\delta_i = \delta_I | \text{Off equilibrium stake choice}) = 1$$

    and the new player will play $D$. This is optimal for will not effect any of $j's$ other relationships because On the other hand an off-equlibrium decrease

- *Choice of links*
  - If impatient players choose to mimick patient players by connecting to two players who know each other in the first period of interaction in the stage game they will cooperate with the first player and defect on the second player. In the second period of interaction they will only have the opportunity to interact with the first of the players since the relationship with the second player is destroyed when the impatient player defects in the first period. However by the start of the second period this first player will hold beliefs $\Pr(\text{impatient}) = 1$ about the impatient player because they will have received a warning about him/her in the

previous period from the second player whom the impatient player played defect against. In the second period of interaction the impatient player and the first player will both play defect. Therefore the most an impatient player can obtain by mimicking a patient player is $z_H + x_H$. It will not be optimal for the impatient player to do this if the following signaling condition holds

$$2x_L > z_H + x_H.$$

The lefthand side is the payoff from successfully deviating on two opponents in a low stakes game and the right is the payoff from cooperating with the first and deviating on the second in a high stakes game.

- *Choice of whether to send a warning*
  - Sending a warning is costless and does not effect a player's payoffs so is also a best response.

*QED*

# 8 Appendix 2: Description of renegotiation equilibrium and proof of Theorem 6

I will first describe the reneotiation concept which is similar to the renegotiation concept used in Watson (1999). Define the sequence of stakes offered by the probability a high stake game is played $\alpha(t)$. Also define the probability that the new player has defected by time $t$ as $\theta(t)$. A jump alteration $\widehat{\alpha}, \widehat{\theta}$ prescribes from period $t$ what the original regime prescribed from $t + \Delta$. Thus $\widehat{\theta}$ and $\widehat{\alpha}$ are specified such that $\widehat{\theta}(w) = \theta(w + \Delta)$ and $\widehat{\alpha}(w) = \alpha(w + \Delta)$ for all $w > t$. Note that $\widehat{\alpha}$ must be such that the right probability mass betrays at time $t - 1$ so that $\widehat{\theta}(t) = \theta(t + \Delta)$. If such a $\widehat{\alpha}$ exists then the new regime is an equilibrium (incentive-compatible) continuation. A stall alteration at $t$ defined by $\widetilde{\alpha}$ and $\widetilde{\theta}$ prescribes from $t + \Delta$ what the original regime prescribed from time $t$. In between times $t$ and $t + \Delta$, $\widetilde{\alpha}$ is set to preserve the current beliefs of the players about each other over the intervening period. Thus $\widetilde{\alpha}$ and $\widetilde{\theta}$ satisfy $\widetilde{\alpha}(w) = \alpha(w - \Delta)$ and $\widetilde{\theta}(w) = \theta(w - \Delta)$ for every $w \geq t + \Delta$, also $\widetilde{\theta}(w) = \theta(t)$ for every $w \in [t, t + \Delta - 1]$. $\widetilde{\alpha}$ on $[t, t + \Delta - 1]$ is arbitrary however for it to be incentive-compatible no type assigned positive probability at time $t$ can now have an incentive to defect during $[t, t + \Delta - 1]$.

Before the start of each period players may reconsider the continuation of their regime. Players decide by voting for or against an exogeneously given incentive-compatible alteration. The players simultaneously vote for or against the alteration. Additionally I assume that no information is revealed in the renegotiation process about players' types. A regime is abandoned in favour of another if in the case of a jump both players accept it or in the case of a stall at least one player accepts it.

Now consider the following strategies for two friends and suppose they do not share a friend in common. I will assume player 1 is the older player and is patient. Player 2 however is a new player and may be patient or impatient.

Player 1

- *Choice of stake*
  - First period of interaction offer a low stakes game.
  - Second period of interaction mix with probability $\beta$ of a high stakes game.
  - Thereafter mix if in all previous periods the stake was low and $(C, C)$ has been the history of play.
  - Offer a high stakes game when the previous period the stake was high and cooperation has been the history of play.
- *Choice of strategy*
  - Play C

Patient Player 2

- *Choice of strategy*
  - Play C

Impatient Player 2

- *Choice of strategy*
  - When the stake of the game is high
    * Play D
  - When the stake of the game is low
    * Mix with probability $\mu$ of cooperating in the first peiod
    * Play C in all subsequent periods

Beliefs

- The belief of the new player is always

$$\Pr(\text{Patient}) = 1$$

- The belief of the older player before the first period is

$$\Pr(\text{Impatient}) = \frac{\lambda}{[(1-2\lambda)(1-\gamma)+\lambda]}$$

- The belief of the older player when the history includes at least one high stakes game is

$$\Pr(\text{Patient}) = 1$$

- The belief of the older player after a history of only low stake games is

$$\Pr(\text{Impatient}) = \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma)+\mu\lambda]}$$

**Theorem 6** *Suppose* $\lambda \geq \frac{(1-\gamma)\Delta z}{z_L+y_H(1-\delta_P)+2\Delta z(1-\gamma)}$, $\beta = \frac{1}{(x_H-x_L)}\left(\frac{(x_L-z_L)}{\delta_I} - x_L\right)$, $\mu$ *satisfies* $\frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}\frac{z_H}{1-\delta_P} + \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}(-y_H) = \frac{z_L}{1-\delta_P}$, $1-\max\left\{\frac{z_L}{x_L}, \frac{z_H}{x_H}\right\} > \delta_I > \frac{x_L-z_L}{x_H}$ *then* $\exists\underline{\delta}$ *such that these strategies are an alteration proof equilibrium of the two person repeated prisoners dilemma for* $\delta_P \in [\underline{\delta}, 1]$.

Proof of equilibrium:

There is always a $\underline{\delta}$ close enough to 1 such that it is optimal for the patient player to cooperate provided there is a positive probability the other player is also patient. To see this note that $\lim_{\delta\to 1}$ of the left hand side of the following equation

$$\Pr(Patient)\frac{z_L}{1-\delta} - (1-\Pr(Patient))\, y_H \geq \Pr(Patient)\, x$$

is $\infty$ provided $\Pr(Patient) > 0$. Where the left hand side is a lower bound on the payoff from cooperation and the right hand side is the payoff from defecting.

The only way an impatient type is revealed given the equilibrium strategies is through playing defect and thereby destroying the friendship. After the first period this only occurs when a high stakes game is played since both patient and impatient players cooperate in low stakes games from period two onwards. It is only when the players interact in a high stakes game that the older player finds out if the other player is patient or impatient.

In periods $t = 1, ..., \infty$ the impatient player is indifferent between cooperating and defecting when the stake of the game is low. The payoff from defecting this period is the same as cooperating and waiting until the following period

$$x_L = z_L + \delta_I \left[\beta x_H + (1-\beta) x_L\right]$$

$$\Rightarrow \beta = \frac{1}{(x_H - x_L)} \left( \frac{(x_L - z_L)}{\delta_I} - x_L \right)$$

. $0 < \beta < 1$ when $\delta_I > \frac{x_L - z_L}{x_H}$. The impatient player will not prefer to cooperate provided that

$$x_H > \frac{z_H}{1-\delta_I}$$

In periods $t = 2, ..., \infty$ the older player who chooses the stake of the game will be indifferent between offering a high or a low stakes game provided that

$$\Pr(\text{Patient}) \frac{z_H}{1-\delta_P} + (1 - \Pr(\text{Patient}))(-y_H) = \frac{z_L}{1-\delta_P}$$

where

$$\Pr(\text{Patient}) = \frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma) + \lambda\mu]}$$

if the impatient player mixes in period 1 with probability $\mu$ of cooperating and $(1-\mu)$ defecting

$$\frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma) + \lambda\mu]} \frac{z_H}{1-\delta_P} + \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma) + \lambda\mu]}(-y_H) = \frac{z_L}{1-\delta_P}$$

Also if $\mu$ satisfies $\frac{(1-2\lambda)(1-\gamma)}{[(1-2\lambda)(1-\gamma)+\lambda\mu]} \frac{z_H}{1-\delta_P} + \frac{\mu\lambda}{[(1-2\lambda)(1-\gamma)+\lambda\mu]}(-y_H) = \frac{z_L}{1-\delta_P}$ then the older player is indifferent between offering a high and low stakes game in the second period and any future period as long as only low stakes games have been offered previously. For $\mu \in [0, 1]$ to exist then $\lambda \geq \frac{(1-\gamma)\Delta z}{z_L + y_H(1-\delta_P) + 2\Delta z(1-\gamma)}$.

I will now prove the equilibrium is alteration proof.

Delays

Given the repetitive nature of the equilibrium the only times a delay is meaningful is between the first and second period and in the instance when a high stakes game has been played and information has been revealed about the type of the unkown player. Firstly consider the incentives for the unknown player to delay. The sequence of stakes is weakly increasing so both types of the unknown player never want to delay an increase in the stake of the game as both benefit from this. Now consider the incentives of the older player for a delay alteration. A delay between the first and second periods is not strictly preferred since in the second period the older player is indifferent between a low or high stake game. In the sequence of stakes, which are offered by the older player, the first high stakes game which arises as a result of mixing induces information about the unknown player to be revealed. A delay after this high stakes game has been played is not preferred by the older player, since the player knows for sure that the other player is patient since the relationship survived the high stake game in the previous period and will prefer to play high stake games.

Jumps

In this equilibrium only three different jumps can occur from a low stake to the mixed stake, from a low stake to a high stake, and from the mixed stake to a high stake. Jumps ahead from the low stake offered in the first period are not preferred by the older player if the payoff from offering a high stake are worse than a low stake. This will not be the case if $\mu \in [0,1]$ since the only difference between the first and second periods is that the older player places a lower probability on the unknown player being impatient. $\lambda \geq \frac{(1-\gamma)\Delta z}{z_L+y_H(1-\delta_P)+2\Delta z(1-\gamma)}$ guarantees that $\mu \in [0,1]$. The only jumps which are possible are jumps from periods in which the older player is required to mix to periods in which he/she offer a high stake for sure. For this type of jump to be incentive compatible the impatient type must defect for sure in the period immediately prior which is not incentive compatible when $\delta_I > \frac{x_L-z_L}{x_H}$ because the impatient type will prefer to wait.